A comparison of the performance of the least squares collocation and the fast Fourier transform methods for gravimetric geoid determination

by Siphiwe Mphuthi, University of Cape Town

Abstract

The Land Levelling Datum (LLD) is the South African vertical datum based on more than 100-year-old tide gauge measurements of mean sea level (MSL). The SAGEOID10 (Merry, 2009) hybrid geoid model was computed for the purpose of replacing the existing vertical datum. Two gravimetric geoid models were computed using different techniques for evaluation of the Stokes' integral, such as, least squares collocation (LSC) and one dimensional fast Fourier transform (1D-FFT) technique.

The FFT method is more reliable than the LSC method, since it requires less computational time on large datasets than the LSC. The geoid model was computed over the province of Gauteng. The computed geoid models, SiPLSC and SiPFFT, were compared to the SAGEOID10 model with standard deviation of 5,6 cm. The long wavelength component of the computed geoid model was compared to the EGM2008 geopotential geoid model with a standard deviation of 4,2 cm.

Keywords

geoid, vertical datum, associated legendre function, EGM2008, LSC, FFT

Introduction

The geoid, by its definition as the equipotential surface of the Earth's gravity field which on average coincides with the undisturbed mean sea level (MSL), provides the appropriate reference surface for heights. Therefore, traditionally, national and regional height datums were defined with respect to a selected network of tide gauges; and height networks were established by terrestrial techniques such as spirit levelling.

The South African vertical datum, generally referred to as the Land Levelling Datum (LLD) or Primary Levelling Network, and classified as a spheroidal orthometric height system, is based on the determined MSL from tide gauge observations (since early 1900s) at Cape Town, Durban, East London and Port Elizabeth [33]. Although Wonnacott and Merry (2012) state that the period of tide gauge observations for determining MSL at these tide gauges was rather short (1-2 years), no exact information on the length of tide gauge observations is available.

The accelerating growth in the use of the global positioning system (GPS) instruments intensified the requirement for geoid heights that are consistent with the attainable accuracy of the ellipsoidal heights that are determined from the GPS. It is for this transformation, from ellipsoidal heights to orthometric heights that an accurate geoid is required. The relationship between the ellipsoidal (h), orthometric (H) and geoidal heights (N) is shown in Fig. 1 and Equation 1.

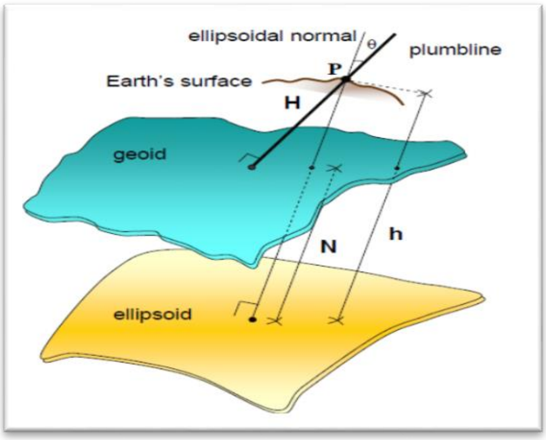

Fig. 1: Relationship between ellipsoidal height (h), orthometric height (MSL, H) and geoid height (N)[27].

$$N = h - H \quad (1)$$

The development and availability of global geopotential models derived from satellite and terrestrial gravity data has led to significant improvement in the accuracy and resolution of global geoids. A sample of recent global geopotential models are as listed in Table 1.

Model	Year	Degree	Data type	Reference
EGM96	1996	360	Satellite(SLR, Altimetry), Terrestrial Gravity	[15]
EIGEN-CG03C	2005	360	CHAMP, GRACE, Terrestrial Gravity, Altimetry	[8]
EGM2008	2008	2190	GRACE, Terrestrial Gravity, Altimetry	[24]
EIGEN-6C4	2014	2190	GRACE, GOCE, SLR, Terrestrial Gravity, Altimetry	[7]
GECO	2015	2190	GOCE + (EGM2008)	[9]
GOCO 05c	2015	720	GRACE, GOCE, SLR, Terrestrial Gravity, Altimetry	[6]

Table 1: Global geopotential models.

Although there has been an improvement in the accuracy and resolution of global geopotential models, and satellite altimetry data, the volume and density of the observed terrestrial gravity data over South Africa remains constant.

However, the computational techniques for evaluating Stokes' integral (for the inner zone contribution) using gravity anomalies derived from the observed terrestrial data has improved, in particular for computational efficiency. The traditional numerical integration is replaced with fast Fourier transform (FFT) and least squares collocation (LSC) techniques.

The improved datasets (geopotential models) and computation techniques provide an opportunity to compute a geoid model over South Africa, to attain geoidal heights and hence orthometric heights at accuracies that are consistent with GPS determined ellipsoidal heights; and orthometric heights that are compatible with South Africa's LLD. This overcomes some of the disadvantages of spirit levelling – labour intensive, expensive, access to remote and mountainous areas.

Several national agencies around the world such as Australia [5], Canada [25] , New Zealand [1], South American countries [2] and the United States of America [25] 1 have embarked on recomputing/redefining their vertical datums; evolving into a geoid-based height system.

In recent times, a continental geoid for Africa, three versions of a regional geoid for southern Africa and a national geoid for South Africa have been modelled. They are partially discussed as follow:

- The African Geoid Project of 2003 (AGP2003), developed by Merry *et al (*2003) is a quasi geoid model which was developed for the purpose of having a unified precise vertical datum for Africa to provide support of infrastructure and development across the continent. The accuracy of the AGP2003 quasi geoid model was only validated over three regions (Algeria, Egypt and South Africa) due to the lack of GPS/levelling observations [19, 20] .
- Three UCT quasi-geoid models were computed for southern Africa, namely, UCT2003, UCT2004 and UCT2006 [18].

- The SAGEOID10 [3] is the South African hybrid geoid which was developed for transforming GPS derived ellipsoidal heights to orthometric heights on the LLD. The precise hybrid geoid model SAGEOID10 was developed for the Chief Directorate: National Geospatial Information (CD: NGI) [3].

The theory of local geoid modelling

It is currently a standard practice to compute the geoidal height at a point by evaluating the attributes in terms of the following components:

$$N = N_L + N_S + N_I, \tag{2}$$

Where:

$N -$ Total geoidal height anomaly at the computation point

$N_L -$ Long wavelength component

$N_S -$ Short wavelength component/inner zone

$N_I -$ Innermost zone component

The WGS84 ellipsoid was used as the reference ellipsoid for geoid modelling in this study. The long wavelength component N_L is computed from the set of spherical harmonic coefficients of the EGM2008 geopotential model truncated at 720 degree and order. The following expression was used for the computation of the long wavelength component [17]:

$$N_L = \frac{GM_g}{\gamma_0(\bar{\varphi})r} . \sum_{n=2}^{n_{max}} \left(\frac{a_g}{r}\right)^n . \sum_{m=0}^{n} \left(\Delta\bar{C}_{n,m} \cos m\lambda \ + \ \bar{S}_{n,m} \sin m\lambda\right) . \bar{P}_{n,m}(\sin\bar{\varphi}), \tag{3}$$

Where:

$GM_g -$ Gravity mass constant of the geopotential model in m^3/s^2 defined from the geodetic model, $\gamma_0(\bar{\varphi}) -$ normal gravity in $m.s^{-2}$

$r -$ Radial distance to the computational point also known as the local elliptic radius in metres m

$a_g -$ Semi-major axis of the geopotential model

$\bar{C}_{n,m}$ *and* $\bar{S}_{n,m} -$ Fully normalised spherical harmonic coefficients (or Stokes' coefficients) of degree *n* and order *m*

$\bar{P}_{n,m} -$ Fully normalised associated Legendre function

$\bar{\varphi} -$ Geocentric latitude of the computation point

φ *and* $\lambda -$ Geodetic latitude and longitude of the computation point

$\Delta\bar{C}_{n,m} -$ It is the difference between the full harmonic coefficient $\bar{C}_{n,m}$ and the harmonic coefficient generated by the normal gravity field $C^*{}_{n,m}$

It is highly recommended to use the recursion method for computation of the fully normalised Legendre function [13]. This method is numerically stable for higher degree and order, especially for computer programming.

The short wavelength component is computed from the Stokes' functions and the gravity anomalies using the Stokes' integral. There are two techniques used for evaluation of the Stokes' integral, namely FFT and LSC techniques. The Stokes' integral can be expressed as follow [10]:

$$N_S = \frac{R}{4\pi\gamma} \iint_{\sigma} \Delta g . S(\psi) d\sigma \tag{4}$$

Where:

$R -$ Mean radius of the Earth

$\Delta g -$ Gravity anomalies

$\psi -$ Geocentric angle/spherical distance

$d\sigma -$ Is an infinitesimal surface element of the unit sphere σ

$S(\psi) -$ Stokes' function

The Stokes' function can be computed as follow [10]:

$$S(\psi) = \frac{1}{\sin(\frac{\psi}{2})} - 6\sin\frac{\psi}{2} + 1 - 5\cos\psi - 3\cos\psi . \ln\left[\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right]. \tag{5}$$

The one dimensional fast Fourier transform (1D-FFT) was used for the evaluation of the Stokes' integral in this study. The Stokes integral is evaluated using an approximated Stokes' kernel function on the sphere and the terrestrial gravity data. Geoid undulations are computed at all gridded points over the study area. This procedure is explained in details by Vella and Featherstone (1999), and Sideris (2013). The 1D-FFT approach gives exactly the same results as the direct numerical integration and it also requires less computer memory. It handles a large amount of data in one run [27].

The LSC technique is a classical technique used in various applications of geodesy and photogrammetry. This technique can handle heterogeneous observations to estimate gravity field components and their standard errors, such as geoidal heights [22]. A linearised observation equation is created for each and every data point. The LSC technique requires more time to handle a larger amount of observations. The LSC technique uses auto- and cross-covariance functions computed by the covariance propagation from the analytically modelled local covariance function. This is expressed by Equation (6) [21]. This technique is explained in detail by Moritz, (1972).

$$K(P,Q) = \sum_{n=2}^{n_{max}} k_n \left(\frac{R^2}{r_P r_Q}\right)^{(n+1)} \bar{P}_n(\cos\psi_{PQ}) \tag{6}$$

Where:

$\bar{P}_n(\cos\psi_{PQ}) -$ Fully normalised Legendre's polynomial functions

ψ_{PQ} – Spherical distance between two points on a sphere

$r_P r_Q -$ Local ellipsoidal radius of point P *and* Q, respectively

k_n– Coefficient computed from the harmonic coefficients as follow:

$$k_n = \sum_{m=0}^{n} \left(\bar{a}_{nm}^2 + \bar{b}_{nm}^2\right) \tag{7}$$

Where:

$\bar{a}_{nm}, \bar{b}_{nm}$ – are the coefficients of fully normalised harmonics.

The influence of the innermost zone to the geoidal height is computed separately by the following expression [10]:

$$N_I = \frac{r}{\gamma}.\Delta g \tag{8}$$

This is applicable if the central area would be a circle with radius r. The local geoid model is computed using the routine approach known as the remove-compute-restore technique. This approach involves the removal of the gravity anomalies generated by the global geopotential model from the terrestrial gravity anomalies. The restore step involves the summation of the long- and short-wavelength component to generate the geoid height. This approach is explained in detail by Vella and Featherstone (1999).

Data compilation

There are various types of data used for determining a geoid model. The accuracy of the geoid model depends on the quality of the data used, its distribution, density of the data and the source. The data used in this study is described as follow:

- The EGM2008 global geoid model has been developed by the United State National Geo-Spatial Agency (NGA). The EGM2008 is complete to degree and order 2159, and it can be extended to degree 2190 and order 2159 when converted to spherical harmonic coefficients [24]. The development of the EGM2008 geopotential model is explained in detail by Pavlis *et al* (2008).
- The available terrestrial gravity data set from the South African Council of Geoscience is used for computation of the short wavelength component, distributed only within the Gauteng province. It is identical to the data set used by Merry (2009) for computation of the SAGEOID2010 geoid model. There were 1853 gravity data points distributed over the region of Gauteng province.

Results and analysis

The gravimetric geoid models were computed over the region of Gauteng province, bound within $-27°0'0 \leq \varphi \leq$ -29°30'0" and 25°0'0 $\leq \lambda \leq 27°0'0"$. Two geoid models were computed in this study, SiPLSC and SiPFFT, using spherical harmonic coefficients from the EGM2008 geopotential model complete to an order and degree 720. A $0{,}3^o \times 0{,}3^o$ grid was used for computation of the local geoid model over the study area.

The SiPLSC geoid model was computed using the least squares collocation method and the SiPFFT geoid model was computed using the one dimensional FFT method. The statistical results of the two geoid models are as illustrated in the Table 2.

Model	**Min. (m)**	**Max. (m)**	**Mean (m)**	**Std. Dev. (m)**	**RMS (m)**
SiPLSC	20,697	28,946	25,929	1,681	25,983
SiPFFT	20,660	28,910	25,892	1,681	25,946

Table 2: Statistics results of the SiPLSC and SiPFFT geoid models.

It is observed from Table 2 that the two approaches used for evaluating Stokes' integral are consistent. The least squares collocation technique allows the consistent treatment of heterogeneous gravity data. It requires an excessive amount of computer time when working with large datasets. It is a multiple-input-single-output method. Due to the increase of data size and quality, a more reliable and efficient technique is required.

The FFT technique produces the same results as the least squares collocation for geoid model computation. The FFT approach differs from the least squares collocation in that it uses regular grid data and can handle a large amount of data in one run. The FFT technique has become a standard procedure for geoid computation. Fig. 2 depicts a 0,5 m interval contour plot of the SiPFFT geoid model over the study area.

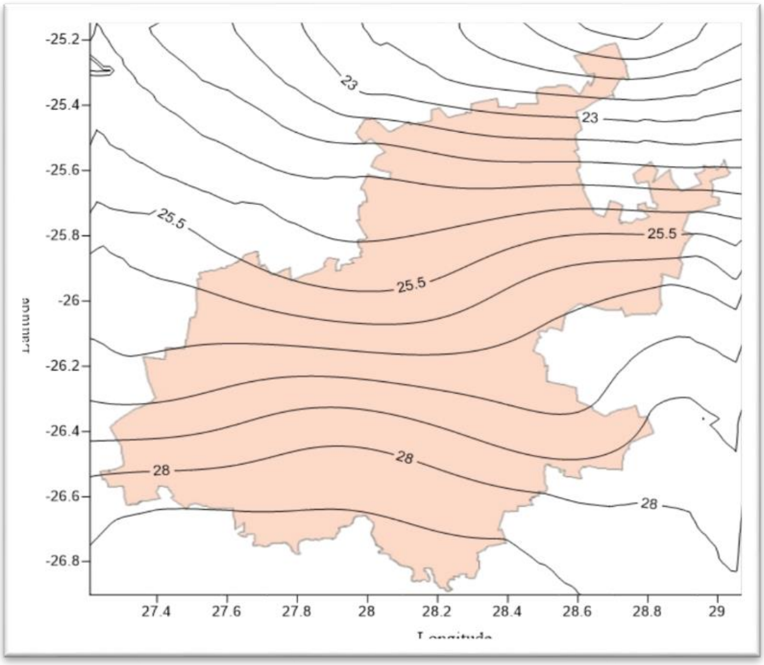

Fig. 2: SiPFFT geoid model at 0,5 m interval contour plot.

The computed geoid model over the region of Gauteng province seems to be gradually increasing towards the south west part of the province. The consistency of the results between the two techniques is illustrated by the statistical results in Table 3.

Comparison	Min. (m)	Max. (m)	Mean (m)	Std. Dev. (m)	RMS (m)
SiPLSC-SiPFFT	0,036	0,037	0,037	0,000	0,037

Table 3: Comparison of the SiPLSC and the SiPFFT geoid model.

The computed geoid models were not compared to GPS/levelling data due to the lack of data over the study area. The comparison of the computed geoid models with the SAGEOID10 hybrid-geoid model over the study area is illustrated by the statistical results in Table 4.

Comparison	Min. (m)	Max. (m)	Mean (m)	Std. Dev. (m)	RMS (m)
SiPLSC- SAGEOID10	-0,254	0,147	-0,039	0,056	0,069
SiPFFT- SAGEOID10	-0,290	0,110	-0,076	0,056	0,094

Table 4: Comparison of the computed geoid models with the SAGEOID10 hybrid geoid model.

The long wavelength component of the computed geoid model was compared to the geoidal height interpolated in a grid of values for the Earth Gravity Model, EGM2008. An online version of the GeoidEval (version 1.46) software was used, available at *(http://geographiclib.sourceforge.net/cgi-bin/GeoidEval)*. The RMS error in the interpolated geoidal height is about 1,0 mm. Only 39 random data points were selected for this comparison. The results of the comparison are illustrated in Table 5.

Comparison	Min. (m)	Max. (m)	Mean (m)	Std. Dev. (m)	RMS (m)
Long wavelength comp. EGM2008	0,133	0,288	0,198	0,042	0,202

Table 5: Comparing the long wavelength component of the computed geoid model to the interpolated geoidal height from the full EGM2008 geopotential model.

The full EGM2008 geopotential model refers to the spherical harmonic coefficients of the EGM2008 model complete to degree and order 2160. The computed long wavelength component is computed using spherical harmonic coefficients of the EGM2008 geopotential model truncated at degree 720. This might have caused the existing discrepancies. This comparison was done to validate the computed long wavelength component.

The existing systematic distortions from the relationship expressed by Equation 1 above can be modelled using a four parametric mathematical model. The mathematical model is used as a surface fit/correction surface to reduce existing distortions between the geometric geoidal height and the gravimetric geoidal height. The existing offset between the local geoid model and the GPS/levelling data can be computed as follow [10]:

$$N^{GPS/levelling} - N^{GM} = h - H - N^{GM} = x_0 + x_1 \cos\varphi \cos\lambda + x_2 \cos\varphi \sin\lambda + x_3 \sin\varphi, \qquad (9)$$

Where:

$N^{GPS/levelling}$ and N^{GM} − Represent a geometric geoidal height and a gravimetric geoidal height

$x_0, x_1, x_2, and\ x_3$ − Represent the four unknown parameters

φ and λ − Geodetic latitude and longitude of the computation point

h and H − Represent the orthometric and the ellipsoidal height

The solution of the unknown parameters can be determined using the least squares method. There are a number of factors which contributes to this offset, a few have been mentioned by Merry (2003).

Conclusions and findings

The effect of using the least squares collocation and 1D-FFT technique for a gravimetric geoid model computation over the region of Gauteng province has been studied in this research.

The geoid solution by the least squares collocation and the 1D-FFT technique differ from each other on average by 3,7 cm. The long wavelength component of the computed geoid model compared to the full EGM2008 geopotential model with a standard deviation of 4,2 cm; only 39 random data points were used for this comparison. This comparison was only done to validate the computed long wavelength component of the computed geoid model.

The 1D-FFT technique is the best geoid determination technique due to an increase in data density and distribution. It handles a large amount of data in one run. The FFT technique has become a standard procedure for geoid computation, particularly for computational efficiency.

The computed geoid model was not corrected for systematic distortions due to the lack of GPS/levelling data over the region of Gauteng province. The GPS/levelling data points and the gravity data need to be intensified to improve the quality of the computed geoid model. This is also necessary to provide improvements on the quality of the vertical datum of South Africa, since the geoid-based vertical datum is defined more by geoid modelling than measurements.

The computed geoid models illustrate the agreement with the SAGEOID10 hybrid geoid model with standard deviation of 5,6 cm. The SiPLSC geoid model seems to be closer to the SAGEOID10 hybrid geoid model than the SiPFFT geoid model.

The SAGEOID10 is the South African hybrid geoid which was developed for transforming GPS derived ellipsoidal heights to orthometric heights on the LLD. However, the accuracy of the orthometric height determined from the conversion depends on the quality of the geoid model and the ellipsoidal height derived from the GPS instrument. The precise hybrid geoid model SAGEOID10 was developed for the Chief Directorate: National Geospatial Information (CD: NGI).

To maintain temporal stability for the local geoid model, it should not be updated too frequently, for instance, immediately after a new model is available. The levelling databases should also be acknowledged since it will still be useful for validating and evaluation of geoid models, and for other scientific studies. The use of the GPS instrument reduces occupation time for surveying techniques.

Acknowledgement

I would like to thank the National Geo-Spatial Information (NGI) for providing data (GPS/Levelling and gravity data) for this research. I'm heartily grateful to my supervisor Dr Ramesh Govind for his encouragement, support and guidance provided from the beginning to the end of the project.

References

[1] M Amos and W Featherstone 2009. Unification of New Zealand's local vertical datums: iterative gravimetric quasigeoid computations. *Journal of Geodesy.* 83(1):57-68.

[2] D Blitzkow, A Matos, G Guimarães, and M Lobianco. 2010. Recent progress of the geoid in South America. *Presentado En La Reunión SIRGAS.*

[3] G Chandler and C Merry. 2010. The South African geoid 2010: SAGEOID10. *Position IT*: 29-33.

[4] SJ Claessens, C Hirt, M Amos, W Featherstone and J Kirby. 2011. The NZGEOID09 model of New Zealand. *Survey Review.* 43(319): 2-15.

[5] W Featherstone, M Filmer, S Claessens, M Kuhn, C Hirt and J Kirby. 2012. Regional geoid-model-based vertical datums–some Australian perspectives. *Journal of Geodetic Science.* 2(4): 370-376.

[6] T Fecher, R Pail and T Gruber. 2015. Global gravity field modeling based on GOCE and complementary gravity data. *International Journal of Applied Earth Observation and Geoinformation.* 35:120-127.

[7] C Förste, S Bruinsma, O Abrikosov, F Flechtner, J Marty, J Lemoine C Dahle and H Neumayer. et al. 2014. EIGEN-6C4-The latest combined global gravity field model including GOCE data up to degree and order 1949 of GFZ Potsdam and GRGS Toulouse. *EGU General Assembly Conference Abstracts.* 3707.

[8] C Förste, F Flechtner, R Schmidt, R König, U Meyer, R Stubenvoll, M Rothacher and F Barthelmes. et al.2006. Global mean gravity field models from combination of satellite mission and altimetry/gravimetry surface data. *Proceedings of the 3rd International GOCE User Workshop, Frascati, Italy,(ESA SP-627.* 6.

[9] M Gilardoni, M Reguzzoni and D Sampietro. 2016. GECO: a global gravity model by locally combining GOCE data and EGM2008. *Studia Geophysica Et Geodaetica*: 1-20.

[10] WA Heiskanen and H Moritz. 1967. Physical geodesy. *Bulletin Géodésique (1946-1975).* 86(1):491-492.

[11] B Hofmann-Wellenhof, H Lichtenegger and E Wasle. 2007. *GNSS–global navigation satellite systems: GPS, GLONASS, Galileo, and more.* Springer.

[12] B Hofmann-Wellenhof and H Moritz. 2005. *Physical geodesy.* Springer.

[13] SA Holmes and WE Featherstone. 2002. A unified approach to the Clenshaw summation and the recursive computation of very high degree and order normalised associated Legendre functions. *Journal of Geodesy.* 76(5): 279-299.

[14] F Lemoine, N Zelensky, D Chinn, D Pavlis, D Rowlands, B Beckley, S Luthcke and P Willis. et al. 2010. Towards development of a consistent orbit series for TOPEX, Jason-1, and Jason-2. *Advances in Space Research.* 46(12): 1513-1540.

[15] FG Lemoine, SC Kenyon, JK Factor, RG Trimmer, NK Pavlis, DS Chinn, CM Cox and SM Klosko. et al. 1998. The development of the joint NASA GSFC and the National Imagery and Mapping Agency (NIMA) geopotential model EGM96.

[16] F Li, J YUE and L ZHANG. 2005. Determination of geoid by GPS/Gravity data. *Chinese Journal of Geophysics.* 48(2): 326-330.

[17] M Losch, and V Seufer. 2003. How to Compute Geoid Undulations (Geoid Height Relative to a Given Reference Ellipsoid) from Spherical Harmonic Coefficients for Satellite Altimetry Applications. *Http://mitgcm.Org/~ mlosch/geoidcookbook/geoidcookbook.Html.*

[18] CL Merry. 2007. Evaluation of global geopotential models in determining the quasi-geoid for Southern Africa. *Survey Review.* 39(305): 180-192.

[19] C Merry. 2003. The African geoid project and its relevance to the unification of African vertical reference frames. *Proceedings of the 2nd FIG Regional Conference.*

[20] C Merry, D Blitzkow, H Abd-Elmotaal, H Fashir, S John, F Podmore and J Fairhead. 2005. A preliminary geoid model for Africa. In *A Window on the Future of Geodesy.* Springer. 374-379.

[21] H Moritz. 1972. *Advanced least-squares methods.* Department of Geodetic Science, Ohio State University Columbus.

[22] H Moritz. 1978. Least-squares collocation. *Reviews of Geophysics.* 16(3): 421-430.

[23] H Moritz. 1980. Advanced physical geodesy. *Karlsruhe: Wichmann; Tunbridge, Eng.: Abacus Press, 1980.* 1.

[24] NK Pavlis, SA Holmes, SC Kenyon and JK Factor. 2008. An earth gravitational model to degree 2160: EGM2008. *EGU General Assembly*: 13-18.

[25] D Roman and ND Weston. 2012. Beyond GEOID12: implementing a new vertical datum for North America. *FIG Proceedings.*

[26] DR Roman and ND Weston. 2012. Beyond GEOID12: Implementing a New Vertical Datum for North America. *Proceedings of the FIG Working Week.* 6.

[27] MG Sideris. 2013. Geoid determination by FFT techniques. In *Geoid Determination.* Springer. 453-516.

[28] W Torge. 2001. *Geodesy.* Walter de Gruyter.

[29] C Tschernig, and D Arabelos. 2005. Computation of a geopotential model from GOCE data using fast spherical collocation—a simulation study. In *A Window on the Future of Geodesy.* Springer. 310-313.

[30] P Ulotu. 2009. Geoid model of Tanzania from sparse and varying gravity data density by the KTH method.

[31] P Vanicek. 1991. Vertical datum and NAVD88. *Surveying and Land Information Systems.* 51(2): 83-86.

[32] J Vella, and W Featherstone. 1999. A gravimetric geoid model of Tasmania, computed using the one-dimensional fast Fourier transform and a deterministically modified kernel. *Geomatics Research Australasia*: 53-76.

[33] R Wonnacott and C Merry. 2011. A New Vertical Datum for South Africa?, *In Conference Proceedings of the AfricaGEO*.

Web-Pages:

1. Zilkoski , 2015 , Retrieved: March 2016, http://gpsworld.com/establishing-orthometric-heights-using-gnss-part-3/
2. National Geodetic Survey (NGS), Retrieved: February 2016, http://www.ngs.noaa.gov/GEOID/GEOID12B/GEOID12B_TD.shtml

Contact Siphiwe Mphuthi, University of Cape Town, mphmat021@gmail.com